Large-scale properties of wave turbulence

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Wave turbulence for systems with only direct (small-scale) turbulent cascades is analyzed at scales much larger than the scale of the pumping. At such scales, the turbulence spectrum is shown to turn into an equilibrium Rayleigh-Jeans distribution with the temperature determined by the pumping scale and energy dissipation rate (the turbulent flux). The behavior of the damping of the waves changes drastically at a scale determined by the mean free path of turbulent waves. Two particular examples of acoustic and capillary-wave turbulence are considered. We also carried out numerics which confirm the theoretical predictions.

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We consider the turbulence of waves with the dispersion law $\omega(k) = k^\alpha$, which allows for the resonant three-wave interactions so that there can be found $\mathbf{k}_1$ and $\mathbf{k}_2$, such that $\omega(\mathbf{k}_1 + \mathbf{k}_2) = \omega(k_1) + \omega(k_2)$. Generally, only energy and momentum are conserved by the interaction. It is well known that an external isotropic pumping acting at some scale k_p^{-1} produces a turbulent energy cascade towards small scales so that the occupation numbers of waves $n(k)$ satisfy a Kolmogorov-like power law at $k \gg k_p$ [1]. The possibility of decay processes $k_p \to k_1 + k_2$ means that the waves with $k < k_p$ should also be excited. The distribution of the long waves and their damping is the subject of the present paper.

A general theory is presented along with two examples of the most physical interest: acoustic and capillary waves. The acoustic dispersion law with $\alpha = 1$ is a marginal one. For $\alpha < 1$, resonant three-wave interactions are forbidden. This means that acoustics have some peculiarities since resonant three-wave processes are possible only for small angles. Nevertheless, they are not forbidden due to the finite damping, which leads to broadening corresponding dynamical structure functions. We show that the broadening is generally a more essential effect than deviations of the dispersion law $\omega(k)$ from its linear form (see also [2]).

Since we consider an isotropic wave system with the energy being a single integral of motion, an inverse turbulent cascade is impossible. It is thus natural that the distribution at $k \ll k_p$ is shown below to be generally an equilibrium Rayleigh-Jeans distribution. Turbulence plays the role of a small-scale noise that sustains the equilibrium. The effective temperature of the equilibrium could be expressed via the characteristics of turbulence. We also show that the interaction of long waves with the short ones from the turbulent spectrum gives the main contribution into the damping of long waves. We consider moderate pumping so that the turbulence at $k \sim k_p$ is assumed to be weakly nonlinear. We show that there exists a crossover scale k_*^{-1} (much larger than the pumping scale) determined by the condition that $\omega(k_*)$ is of the order of damping of waves at the pumping scale. For $k \gg k_*$, the standard formalism of the kinetic equation is shown to be applicable for $\alpha > 1$ so that both the equilibrium distribution and the damping decrement can be readily found. These results can be extended to the acoustic case $\alpha = 1$. The effective temperature characterizing the long-scale equilibrium is proportional to the square root of the turbulent flux P (the energy dissipated per unit volume per unit time). For $k \ll k_*$, one should take into account high-order nonlinear corrections. The waves with $k \ll k_*$ should be in thermal equilibrium with the same effective temperature, yet the detailed theoretical description (including the k dependence of the damping) is more difficult. It will be briefly discussed at the end of the analytical part of the paper.

We use the description of a wave system in terms of the normal variables a_k, which satisfy equations [1]

$$i\frac{\partial a_k}{\partial t} = \omega_k a_k + \frac{\delta H_{\rm int}}{\delta a_k^*} + f_k - i\gamma_{0k} a_k\,, \tag{1}$$

$$H_{\rm int} = \frac{1}{2}\int V_{123} a_1^* a_2 a_3 (2\pi)^d \delta(\mathbf{k}_2 + \mathbf{k}_3 - \mathbf{k}_1) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 + \frac{1}{6}\int U_{123} a_1 a_2 a_3 (2\pi)^d \times\delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) d\vec{k}_1 d\vec{k}_2 d\vec{k}_3 + \text{c.c.}$$

Here γ_{0k} is the bare (linear) damping, d is the dimensionality of space, $d\vec{k} \equiv d^d k/(2\pi)^d$, $a_i = a(k_i)$, and f_k is the pumping force. The coefficients of nonlinear interaction V and U are generally independent. There are two alternative ways to excite turbulence. The first one is to use a random pumping force with a characteristic wave vector k_p. The second way is related to instability of the waves with k_p, which can be described by taking negative $\gamma_0(k)$ at $k = k_p$. Our analytical results do not depend on the way of excitation. The numerics are based on the second scheme. The properties of both turbulence at $k \gg k_p$ and equilibrium at $k \ll k_p$ are universal and do not depend on the way of excitation.

The main objects of the analysis are the pair correlation function $F(t,\mathbf{k})$ and the Green's function $G(t,\mathbf{k})$ determined as the following averages:

$$F(t,\mathbf{k})(2\pi)^d\delta(\mathbf{k}-\mathbf{k}') = \langle a(t,\mathbf{k})a^*(0,\mathbf{k}')\rangle\,, \qquad (2)$$

$$G(t,\mathbf{k})(2\pi)^d\delta(\mathbf{k}-\mathbf{k}') = \langle \delta a(t,\mathbf{k})/\delta f(0,\mathbf{k}')\rangle\,, \qquad (3)$$

where δa designates the response of a solution of (1) to the variation δf of the pumping force. One can develop the standard Wyld diagram technique [3,4] for calculating the objects (2) and (3). This technique enables one to represent (2) and (3) as a series over the interaction vertices V, U introduced by (1). We start by considering the case of the weak nonlinearity and restrict ourselves by the first contributions to (2) and (3), then we shall discuss high-order corrections.

It is covenient to investigate contributions to (2) and (3) in terms of the so-called self-energy function Σ and the mass operator Φ defined as follows:

$$G(q) = \left[\omega - \omega_k + i\gamma_{0k} - \Sigma(q)\right]^{-1}, \qquad (4)$$

$$F(q) = \left[\Phi(q) + \Phi_0(q)\right]|G(q)|^2\,. \qquad (5)$$

Here we passed to Fourier representation in time, and introduced the shorthand notation $q = (\omega,\mathbf{k})$. The real part of Σ determines the correction to ω_k due to interaction, the imaginary part $\gamma = -\mathrm{Im}\,\Sigma$ determines the damping of the waves, Φ_0 is the pair correlator of the pumping force f, and Φ has the meaning of the renormalized pumping. The first (so-called one-loop) contributions to Σ and Φ read

$$\Sigma(q) = -\bigg(\int |V_{k12}|^2 G_{q_1}F_{q_2}\delta(q-q_1-q_2)dq_1dq_2 + \int |V_{1k2}|^2(G_{q_1}F_{q_2} + G^*_{q_2}F_{q_2}) \times\delta(q-q_1+q_2)dq_1dq_2 + \int |U_{k12}|^2 G^*_{q_1}F_{q_2}\delta(q+q_1+q_2)dq_1dq_2\bigg)\,, \qquad (6)$$

$$\Phi(q) = \int |V_{1k2}|^2 F_{q_1}F_{q_2}\delta(q-q_1+q_2)dq_1dq_2 + 1/2\int |V_{k12}|^2 F_{q_1}F_{q_2}\delta(q-q_1-q_2)dq_1dq_2 + \int |U_{k12}|^2 F_{q_1}F_{q_2}\delta(q+q_1+q_2)dq_1dq_2\,. \qquad (7)$$

Here dq designates $d\omega d^dk(2\pi)^{-d-1}$ and, say $\delta(q)$, means the product $(2\pi)^{d+1}\delta(\omega)\delta(\mathbf{k})$.

The expressions (6) and (7) together with (4) and (5) can be considered as a closed system of one-loop equations for the objects (2) and (3). It is convenient to introduce the function n:

$$F(q) = -(1/\pi)n(q)\mathrm{Im}G(q)\,, \qquad (8)$$

having the meaning of the occupation numbers of waves. For weak turbulence, the factor $\mathrm{Im}G(q)$ has a sharp maximum near $\omega = \omega_k$. That means that for $\alpha > 1$, one can substitute $n(q)$ by $n(k) = n(\omega_k,\mathbf{k})$ into (6) and (7). Then, using the sum rule $\int d\omega\,\mathrm{Im}G(q) = -\pi$, one can derive from (6) and (7) the standard kinetic equation. The case of acoustic waves ($\alpha = 1$) deserves special consideration: the substitution $n(q) \to n(k)$ is incorrect in this case since both $n(q)$ and $\Sigma(q)$, treated as functions of $\omega - \omega_k$, have the same characteristic scale, which is $\gamma = -\mathrm{Im}\Sigma$. Nevertheless, both at $\alpha > 1$ and at $\alpha = 1$, a solution of the diagrammatic equations for n in the region $k \gtrsim k_p$ is the Kolmogorov-like turbulent spectrum with the energy flux P. Below we will be interested in the behavior of (6) and (7) at $k \ll k_p$.

For small k, the main contribution to the integrals (6) and (7) is determined by the region $k \sim k_p$. That means that if $k \gg k_*$ (where ω_{k_*} is of the order of the damping of waves with $k \sim k_p$), then for $\alpha > 1$ we can neglect the width of G in (6) and (7) and to substitute $n(q) \to n(k)$, $\mathrm{Im}G(q) \to -\pi\delta(\omega-\omega_k)$. After the substitution, we come to the standard expressions that follow from the kinetic equation for waves [1]

$$\gamma(q) = -\mathrm{Im}\Sigma = \pi^2\int |V_{2k1}|^2(n_1-n_2)\delta(\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}) \times\delta(\omega_{k_2}-\omega-\omega_{k_1})d\vec{k}_1d\vec{k}_2\,, \qquad (9)$$

$$\Phi(q) = \frac{\pi^2}{2}\int |V_{2k1}|^2 n_2n_1\delta(\mathbf{k}_2-\mathbf{k}_1-\mathbf{k}) \times\delta(\omega_{k_2}-\omega_{k_1}-\omega)d\vec{k}_1d\vec{k}_2\,, \qquad (10)$$

where $n_i = n(k_i)$. To characterize the damping and spectral density of the waves, we can substitute here $\omega \to \omega_k$. The above derivation is not precisely correct for the acoustics, since in this case the argument of $\delta(\omega_{k_2} - \omega_{k_1} - \omega_k)$ is equal to zero only for the zero angle between $\mathbf{k}_1$ and $\mathbf{k}_2$. That means that one should take into account the concrete form of $\mathrm{Im}G$ as a function of $\omega - \omega_k$, which leads to a complicated dependence of both $\gamma(k)$ and $\Phi(k)$ on $\omega - \omega_k$. In three dimensions, nevertheless, the expressions (9) and (10) can be successfully used for evaluating long-scale values of γ and Φ and consequently for extracting their k dependence. For this, one can take in (9) and (10) as n_i the values of $n(q)$ again at $\omega = \omega_k$ and to treat an integral over angles like $\int d\theta\,\theta\delta(A(1-\cos\theta))$ as $(2A)^{-1}$. One can check that in the region $k_* \ll k \ll k_p$, high-order contributions into Φ and γ can be neglected. Indeed, those contributions contain additional dimensionless factors of the type $\int V_{k12}V_{k'13}G_{q_1}G_{q_3}F_{q_2}\delta(q-q_1-q_2)\delta(q'-q_1-q_3)dq_1dq_2dq_3$, which are of the order of γ/ω (only in three dimensions where there is enough angle integrations) [5].

Since the steady state is given by $n(k) = \Phi(k)/\gamma(k)$, then it is evident from (9) and (10) that the contribution into $\gamma(k)$ from $k_1 \gg k$ will contain the extra factor $\omega(k)$ as compared to $\Phi(k)$. Therefore, $n(k) \propto \omega^{-1}(k)$, which is the equilibrium Rayleigh-Jeans distribution.

Let us consider the interaction vertex V having the homogeneity index m and asymptotic behavior at $k \ll k_1$ as follows:

$$|V_{k_1kk_2}|^2 \propto k^{m_1}k_1^{2m-m_1}\,, \qquad (11)$$

$\mathbf{k}_2 = \mathbf{k} - \mathbf{k}_1$ being assumed. Steady Kolmogorov-like turbulence spectrum $n(k_1) \propto P^{1/2}k_1^{-m-d}$ exists at $k_1 \gtrsim k_p$ if the locality condition $m_1 > m + 2 - 2\alpha$ is satisfied [1]. One thus gets for $k_* \ll k \ll k_p$

$$\gamma(k) \propto P^{1/2} k^{m_1+\alpha-1} k_p^{m+1-2\alpha-m_1}\,, \tag{12}$$

$$\Phi(k) \propto P k^{m_1-1} k_p^{1-\alpha-m_1-d}\,, \tag{13}$$

where as previously $\gamma(k) = \gamma(\omega_k, k)$, $\Phi(k) = \Phi(\omega_k, k)$. A steady distribution could be found from $I_k = -\gamma(k)n(k) + \Phi(k) = 0$:

$$n_k \propto k^{-\alpha} k_p^{\alpha-m-d} P^{1/2}\,. \tag{14}$$

This is an equilibrium Rayleigh-Jeans distribution $n_k = T/\omega_k$ with the effective temperature related to the energy flux P carried by the turbulent waves

$$T_{\text{eff}} \propto k_p^{\alpha-m-d} P^{1/2}\,. \tag{15}$$

Note that the distribution also turns into zero the part of the collision integral that corresponds to the interaction with $k_1 \simeq k$.

For capillary waves, $\omega_k = (\sigma/\rho)^{1/2} k^{3/2}$, where σ is the coefficient of the surface tension and ρ is the density of the liquid. The dimensionality for capillary waves is $d = 2$. The interaction coefficient V is as follows:

$$V_{k12} = \pi^{5/2}(\sigma/\rho)^{1/4}(f_{k12} + f_{k21} - f_{12k})\,,$$

where

$$f_{k12} = (kk_1/k_2)^{1/4}[(k-k_1)^2 - k_2^2]\,.$$

We see that $\alpha = 3/2, m = 9/4$, and $m_1 = 5/2$. Nevertheless, for $k \gg k_*$ one should use the value $m_1 = 7/2$ since there occurs a cancellation of the main contribution to V_{k12} on the resonance surface where $\omega_k + \omega_1 = \omega_2$. Thus, the Kolmogorov-like spectrum is [1] $n_k = \lambda P^{1/2} k^{-17/4}$ with $\lambda \simeq 8\pi(4\rho^3/\sigma)^{1/4}$ [1], which leads to $k_* \sim P^{1/2} k_p^{1/4} \left(\rho/\sigma^3\right)^{1/4}$. For $k_* \ll k \ll k_p$, we get

$$T_{\text{eff}} \sim (\sigma/\rho)^{1/2} \lambda P^{1/2} k_p^{-11/4}\,,$$

$$\gamma \sim \rho^{5/4} P^{1/2} \sigma^{-3/4} k_p^{-13/4} k^4\,.$$

For acoustics $\omega_k = ck$,

$$V_{k12} = \sqrt{ckk_1k_2/\rho}\,(3g + \cos\theta_1 + \cos\theta_2 + \cos\theta_{12}),$$

where g is of the order of unity [1]. One thus gets $m = 3/4$ and $m_1 = 1$. For $d = 3$, we find $k_* \sim \sqrt{Pk_p/(\rho_0 c^3)}$. Then one can obtain

$$T_{\text{eff}} \sim \sqrt{P\rho_0 c/k_p^7}\,,$$

$$\gamma(k) \sim \sqrt{P/(\rho_0 c k_p)}\,k\,.$$

Let us now consider the largest scales with $k \ll k_*$. The expressions (6) and (7) determined by the first diagrams give in this limit

$$\gamma(q) \sim \int \frac{|V_{k_1+kkk_1}|^2}{\gamma_{k_1}} \left(\frac{\partial n_{q_1}}{\partial k_1} \frac{\mathbf{k}\mathbf{k}_1}{k_1} + \frac{\partial n_{q_1}}{\partial \omega_1}\omega \right) d\vec{k}_1, \tag{16}$$

$$\Phi(k) \sim \int \frac{|V_{k_1+kkk_1}|^2}{\gamma_{k_1}} n_{q_1}^2 \, d\vec{k}_1\,. \tag{17}$$

Here we used (8) and estimated the integral over frequences on the basis of (4). After substituting $n(q)$ corresponding to the Kolmogorov spectrum into (16) and (17), we conclude that $\gamma \propto k^{2m_1+\alpha}$ and that γ is P independent and that $\Phi \propto \sqrt{P} k^{2m_1}$. For $k \ll k_*$, we cannot neglect high-order contributions to γ, Φ. Let us take, e.g., the fourth-order contribution to Φ. Using the same ideas, we obtain after integration over frequencies

$$\Phi^{(4)} \sim \int d\vec{k}_1\, d\vec{k}_2\, \Upsilon(k_1, k_2, k) \frac{n^3(k_1)}{\gamma^2(k_1)},$$

where Υ is some function independent on P, the asymptotic behavior of which is determined by the two border vertices of the diagram and is $\propto k^{2m_1}$ as a consequence of (11). Thus, we conclude that $\Phi^{(4)} \propto \sqrt{P} k^{2m_1}$, which means that $\Phi^{(4)}$ is of the order of (17). The same story is with all high-order contributions to Φ and γ: All the contributions have the same k dependence and are of the same order. Most probably that means that Φ and γ can be estimated using the first contributions (16) and (17). One may thus assume that the equilibrium waves with the effective temperature (15) that exist at largest scales have $\gamma/\omega_k \propto k^{2m_1}$.

To see how a large-scale equilibrium spectrum turns into a small-scale turbulent cascade, we carried out the numerical simulation solving isotropic kinetic equation for acoustic and capillary waves:

$$\frac{\partial n_i}{\partial t} = I_i + \Gamma_i n_i - 2n_i \sum_{j=L}^{L+i} W(j,i) n_{j-i}. \tag{18}$$

Here $i = \omega_k/\omega_1$ is the mode number, L is the total number of modes, and Γ_i represents an instability growth rate. The last term is a nonlinear damping that models the absorption of waves by the region $i > L$ and provides for an effective sink for the cascade (see [1] for details).

In order to show that our results are resolution independent, we performed numerics for a variety of L. In our simulations, Γ_i was chosen in the form of a equilateral triangle with the width $\Delta\omega$ and with the top at $i_0 = \omega_0/\omega_1 = L/2$, the maximum of Γ_i is $\Gamma = \Gamma(i_0)$. The thermal noise was used as the initial value of variable n_i in all regions $i \le L$; its level was changed in our simulations within the interval $[10^{-4}$–$5\times10^{-2}]$. We set $n_i(t) \equiv 0$ at $i > L$. The time derivative was approximated by the first-order finite difference scheme. The time step was sufficiently small (5×10^{-4}–10^{-3}) to provide stability of the numerical procedure.

Our numerics show that both for capillary and acoustic waves, the evolution has two well-defined stages: during the first (fast) one, the steady turbulent distribution is formed in the inertial interval $L/2 < i < L$, then the steady quasiequilibrium distribution slowly appears at $i \ll L$. The exponent of the steady distribution

$$S_i = \frac{\log(n_{i+1}/n_i)}{\log[i/(i+1)]}$$

is shown in Fig. 1 as a function of $\log(i)$ at $i < L/2$ for $L = 1000$. The equilibrium distribution with $S_i = 1$

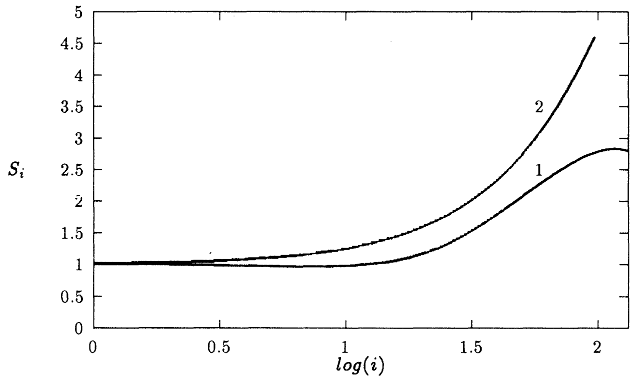

FIG. 1. Local index $S(i)$ vs $\log_{10}(i)$ for steady distributions. For capillary waves, curve 1 ($i_0 = 500$, $\Delta\omega/\omega_0 = 1$, $\Gamma = 100$). For acoustic waves, curve 2 ($i_0 = 500$, $\Delta\omega/\omega_0 = 8/5$, $\Gamma = 100$).

is distorted with the increasing of $\log(i)$ and S_i reaches eventually the Kolmogorov values (9/2 for acoustics and 17/6 for capillary waves).

The prediction for the temperature could be reexpressed solely in terms of the pumping characteristics by using the relation $P^{1/2} \simeq \Gamma b \lambda k_p^{\alpha - m}$ [1], $\omega_0 \equiv \omega_{k_p}$. One thus gets

$$T \simeq \Gamma (b\lambda)^2 k_p^{2\alpha - 2m - d} \simeq \Gamma \lambda^2 b^{(2m+d)/\alpha} \omega_0^{(2\alpha - 2m - d)/\alpha} .$$

For capillary waves, that gives

$$T \simeq \Gamma (\sigma^4/\rho)^{1/3} k_p^{-7/2} \simeq \Gamma (\sigma^5/\rho^2)^{1/3} \omega_0^{-7/3} .$$

The simulations with i_0 being 300, 400, and 500 at constant $\Delta\omega$ and Γ confirm the dependence $T(\omega_0)$. For the steady turbulence under $\Delta\omega = \omega_0$, $\Gamma = 100$, we found the following dependence of the temperature on the pumping frequency:

$$T(300)/T(400) = 1.95 \approx (3/4)^{-7/3} = 1.95 ,$$
$$T(400)/T(500) = 1.68 \approx (4/5)^{-7/3} = 1.68 .$$

For acoustics, the theoretical prediction is

$$T \simeq \Gamma (\rho c_s^3)^{1/2} k_p^{-4} \simeq \Gamma (\rho c_s^{11})^{1/2} \omega_0^{-4} .$$

Numerically, we obtained the following ratios of temperatures for steady spectra under $\Delta\omega/\omega_0 = 8/5$, $\Gamma = 20$:

$$T(300)/T(400) = 3.12 \approx (3/4)^{-4} = 3.16 ,$$
$$T(400)/T(500) = 2.46 \approx (4/5)^{-4} = 2.44 .$$

The linear dependence $T(\Gamma)$ is also confirmed by numerics.

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